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HEAT TRANSFER BY HARTMANN'S FLOW IN THERMAL ENTRANCE REGION

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Abstract—This paper deals with a theoretical analysis to determine the heat-transfer characteristics in the thermal entrance region, in which an electrically conducting fluid flows in established laminar flow between two parallel plates with wall heat transfer under the influence of a uniform transverse magnetic field, so-called Hartmann's flow. The conditions of wall heat transfer treated in this paper are both the case of prescribed uniform wall heat flux and that of uniform wall temperature. The internal heat generation by Joule heating is taken into account in this analysis but the viscous dissipation is neglected. The analytical method is analogous to that used in authors' another paper.

The heat-transfer characteristics of entrance region approach those of fully developed situation which were analysed by Siegel. The heat transfer by laminar flow of a non-conducting fluid which was analysed as a special case coincides well with those by Sellars and others.

These numerical calculations were done by means of KDC-1 (Kyoto University Digital Computer-1).

NOMENCLATURE

a, one half of channel width;
B_o, density of magnetic flux;
c_p, specific heat at constant pressure;
C_n, coefficient in series expansion of temperature;
D_n, coefficient in series expansion of temperature;
E_n, coefficient in series expansion of temperature;
F_n, coefficient in series expansion of temperature;
h, heat-transfer coefficient;
j_z, circulating current density generated within the fluid;
K, Hartmann number;
k, thermal diffusivity;
M, parameter defined by equation (30);
Nu, Nusselt number;
Pr, Prandtl number;
q, heat flux at wall;
Re, Reynolds number;
T, temperature;
u, velocity of fluid;
u_m, mean velocity of fluid;
u_{max}, velocity at the centre axis, $y = 0$;
x, co-ordinate;
Y_n, eigenfunction;
Y'_n, derivative of eigenfunction ($= \mathrm{d}Y_n/\mathrm{d}\eta$);
y, co-ordinate.

Greek symbols

α, pressure gradient along the channel;
β_n, eigenvalue;
γ_n, eigenvalue;
η, non-dimensional co-ordinate ($= y/a$);
λ, thermal conductivity
ν, kinematic viscosity;
ξ, non-dimensional co-ordinate;
ρ, density;
σ, electrical conductivity;
ψ_n, eigenfunction;
ψ'_n, derivative of eigenfunction ($= \mathrm{d}\psi_n/\mathrm{d}\eta$).

1. INTRODUCTION

IN THE past several years, there have been many researches on the convective heat transfer of electrically conducting fluid under the influence of magnetic field passing through the channel, duct or plate, because of the necessity of the planning of magneto-hydrodynamic generator, electromagnetic flow meter and so on [1–4].

The established laminar flow of a uniform conducting incompressible fluid between two

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parallel plates under a uniform magnetic field which is imposed perpendicular to the bounding walls was studied by Hartmann and Lazarus and was summarized by Cowling [5]. This flow is so-called Hartmann's flow, and the velocity profile is flattened from parabolic pattern.

The heat-transfer characteristics by Hartmann's flow have been theoretically investigated by Nigam *et al.* [3] or Siegel *et al.* [4] for the thermal entrance region. Nigam *et al.* treated only a case of constant wall temperature which was suddenly changed from an another different constant wall temperature at certain place, and Siegel *et al.* analysed the wall temperature distribution of thermal entrance region under constant heat flux at wall. But, if there exists the internal heat generation within the fluid, the temperature distribution in the fluid has a very curious pattern, and moreover the relation between local Nusselt number and non-dimensional distance from the inlet is different from ordinary non-internal heat generating fluid, as shown in the previous paper [6]. This tendency will be emphasized owing to the non-uniform Joule heating in the conducting fluid which flows with Hartmann's velocity profile.

In this paper, the temperature distribution within the fluid and the local Nusselt number in the thermal entrance region are evaluated by the analogous method to the previous paper [6], for several kinds of wall heat transfer and for different values of Hartmann number.

2. TEMPERATURE DISTRIBUTION

In this paper, the following assumptions are made:

(1) The flow is an established Hartmann's flow.
(2) The heat produced by viscous force is neglected.
(3) The heat conduction in x-direction is neglected.
(4) The physical properties of fluid are independent of temperature and are constant.

As shown in Fig. 1, the fluid flows from left to right in x-direction between two plates, $y = a$ and $y = -a$, with Hartmann's velocity profile under the influence of uniform magnetic field imposed in y-direction. By using the assumption (1), the relation between u, u_m and $u_{\max}$ are given as follows [5]:

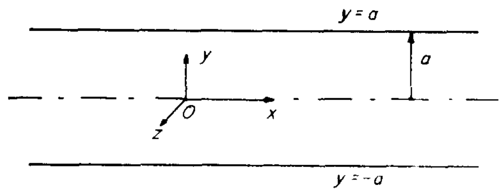

FIG. 1. Co-ordinate.

$$\frac{u}{u_m} = \frac{\cosh K - \cosh (K\eta)}{\cosh K - (1/K) \sinh K}, \tag{1}$$

$$\frac{u}{u_{\max}} = \frac{\cosh K - \cosh (K\eta)}{\cosh K - 1}, \tag{2}$$

where K is Hartmann's number, and it is given by

$$K = \sqrt{\left(\frac{\sigma}{\rho\nu}\right)}\, aB_o \tag{3}$$

In order to derive this velocity profile, it is assumed that the total electric current, $\int j_z \, dy$, flowing between $y = -a$ and $y = a$ vanishes.

In Fig. 1, the thermal entrance region starts from $x = 0$ and there exist both wall heat transfer and Joule heating in the region of $x > 0$, then the equation of heat flow may be as follows:

$$u \frac{\partial T}{\partial x} = k \frac{\partial^2 T}{\partial y^2} + \frac{1}{\rho c_p} \frac{j_z^2}{\sigma}, \tag{4}$$

where j_z^2/σ is the internal heat source due to the electrical dissipation and the circulating current density j_z is given by

$$j_z = \frac{\alpha}{B_o} \left[1 - \frac{K \cosh (K\eta)}{\sinh K}\right]. \tag{5}$$

2.1. *Uniform heat flux at wall*

The problem can be separated into the following two simpler cases:

(1) A case having Joule heating with insulating wall;
(2) A case having no internal heat generation due to Joule heating but uniform heat transfer q at wall.

If the temperatures are defined by T_Q and T_q, respectively, the equations for problem (1) and (2) are as follows:

(1)
$$u\frac{\partial T_Q}{\partial x}=k\frac{\partial^2 T_Q}{\partial y^2}+\frac{1}{\rho c_p}\frac{j_z^2}{\sigma}. \tag{6}$$

Boundary conditions:

$$x=0:\ T_Q=0,\ y=0:\ \partial T_Q/\partial y=0,\ y=a:\ \partial T_Q/\partial y=0.$$

(2)
$$u\frac{\partial T_q}{\partial x}=k\frac{\partial^2 T_q}{\partial y^2}. \tag{7}$$

Boundary conditions:

$$x=0:\ T_q=0,\ y=0:\ \partial T_q/\partial y=0,\ y=a:\ \partial T_q/\partial y=-q/\lambda.$$

The solution T of equation (4) can be written as follows, by virtue of linear equation:

$$T=T_Q+T_q. \tag{8}$$

In this section, the fluid temperature is defined as the excess temperature over the value at $x=0$.

2.1.1. *Insulated wall with Joule heating within the fluid.* Consider the fully developed situation. If the fluid temperature is designated by $T_{Q\infty}$ in this situation, $T_{Q\infty}$ becomes

$$u\frac{\partial T_{Q\infty}}{\partial x}=k\frac{\partial^2 T_{Q\infty}}{\partial y^2}+\frac{1}{\rho c_p}\frac{j_z^2}{\sigma}. \tag{9}$$

Using equation (9) and the mixed-mean temperature T_m,

$$T_m=\frac{\int_{-a}^{a}Tu\,\mathrm{d}y}{\int_{-a}^{a}u\,\mathrm{d}y}, \tag{10}$$

the following equation can be obtained:

$$\frac{\partial T_{Q\infty}}{\partial x}=\frac{\partial T_m}{\partial x}=\frac{1}{au_m c_p\rho}\left(\int_0^a\frac{j_z^2}{\sigma}\mathrm{d}y\right). \tag{11}$$

Therefore, by using equations (3) and (5), $T_{Q\infty}$ is obtained as follows:

$$T_{Q\infty}=\frac{\alpha^2 a^2 x}{u_m\rho^2 c_p\nu}\left[-\frac{1}{K^2}+\frac{1}{2K\sinh^2 K}\left(K+\frac{1}{2}\sinh 2K\right)\right]+F(y), \tag{12}$$

where $F(y)$ is a function of y alone and shows the temperature profile of fully developed situation. Substitution of equation (12) into equation (9) gives $F(\eta)$, that is

$$F(\eta)=\frac{(RePr)^2\nu k}{a^2 c_p}\left\{\left(\frac{\sinh K}{\cosh K-(1/K)\sinh K}\right)^2\left(\left[-1+\frac{K}{2\sinh^2 K}\left(K+\frac{1}{2}\sinh 2K\right)\right]\left[\frac{(\eta^2/2)\cosh K-(1/K^2)\cosh(K\eta)}{\cosh K-(1/K)\sinh K}\right]-\left\{\frac{\eta^2}{2}-\frac{2\cosh(K\eta)}{K\sinh K}+\frac{K^2}{2\sinh^2 K}\left[\frac{\eta^2}{2}+\frac{1}{4K^2}\cosh(2K\eta)\right]\right\}\right)+C_o\right\}. \tag{13}$$

In order to determine the temperature in the entrance region, it is convenient to define a temperature T_Q^+ as follows:

$$T_Q=T_{Q\infty}+T_Q^+. \tag{14}$$

Then, equation (6) yields

$$u\frac{\partial T_Q^+}{\partial x}=k\frac{\partial^2 T_Q^+}{\partial y^2}. \tag{15}$$

Boundary conditions:

$$x=0:\ T_Q^+=0,\ y=0:\ \partial T_Q^+/\partial y=0,\ y=a:\ \partial T_Q^+/\partial y=0.$$

By using the following Reynolds number Re and Prandtl number Pr,

$$Re=u_m a\rho/\mu,\quad Pr=\mu/(\rho k), \tag{16}$$

the solution of equation (15) can be expressed as follows:

$$T_Q^+=\frac{(Re\,Pr)^2\nu k}{a^2 c_p}\sum_{n=1}^{\infty}C_n Y_n\exp\left(-\beta_n\frac{u_m}{u_{\max}}\xi\right) \tag{17}$$

$$\xi=\frac{1}{Re\,Pr}\frac{x}{a}. \tag{18}$$

And, β_n and Y_n are eigenvalues and eigenfunctions, respectively, of the following Sturm–Liouville type differential equation,

$$\frac{d^2Y_n}{d\eta^2} + \beta_n \frac{u}{u_{\max}} Y_n = 0. \qquad (19)$$

Boundary conditions:

$\eta = 0$: $Y_n' = dY_n/d\eta = 0$, $\eta = 1$: $Y_n' = 0$.

From the condition of $T_Q^+ = 0$ at $x = 0$ in equation (15), the coefficients C_n are

$$\left.\begin{aligned} C_n &= -\frac{\int_0^1 u \frac{F(\eta) - C_o}{(Re\,Pr)^2 \nu k/(a^2 c_p)} Y_n \, d\eta}{\int_0^1 u Y_n^2 \, d\eta} \\ C_o &= -\frac{\int_0^1 u \frac{F(\eta) - C_o}{(Re\,Pr)^2 \nu k/(a^2 c_p)} \, d\eta}{\int_0^1 u \, d\eta}. \end{aligned}\right\} \qquad (20)$$

These numerical calculations were done by the digital computer KDC-1 (Kyoto University Digital Computer-1) [6].

Table 1. Eigenvalues and eigenfunctions at wall

	n	Authors' values β_n	Authors' values $Y_n(1)$	Siegel's values β_n'	Siegel's values $Y_n(1)$
$K = 0$	1	18·4986	−1·2708	18·3083	−1·2697
	2	69·4087	1·4035	68·9517	1·4022
	3	152·5951	−1·4931	151·551	−1·4916
	4	268·0768	1·5612	266·163	1·5601
	5	416·6615	−1·6241	412·783	−1·6161
$K = 4$	1	14·8752	−1·1528	14·7866	−1·1520
	2	56·8958	1·2579	56·5509	1·2569
	3	125·7058	−1·3304	124·9158	−1·3294
	4	221·3816	1·3868	219·8498	1·3856
	5	341·3523	−1·4285	341·3385	−1·4319
$K = 8$	1	12·5923	−1·0660		
	5	49·0872	1·1448		
	3	108·9721	−1·2037		
	4	192·2326	1·2505		
	5	410·5202	−1·3712		

The eigenvalues, β_n, and eigenfunctions at wall, $Y_n(1)$, are tabulated in Table 1. In order to compare our results with Siegel's values [4], their results multiplied by $(2/3)(u_{\max}/u_m)$ are shown in this table [Appendix], since equation (19) is different from the equation used by Siegel and others. In this table, our values agree well with Siegel's values.

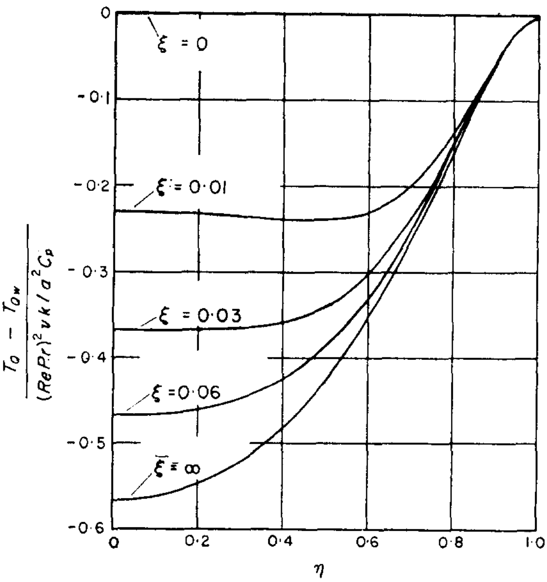

FIG. 2. Fluid temperature with Joule heating and no heat flow through the wall ($K = 4$).

Fig. 2 shows the fluid temperature distribution for $K = 4$, as one example, at various positions, ξ.

2.1.2. *Uniform wall heat flux without Joule heating.* Let us consider the fully developed region having a defined temperature, $T_{q\infty}$. Equation (7) becomes

$$u \frac{\partial T_{q\infty}}{\partial x} = k \frac{\partial^2 T_{q\infty}}{\partial y^2}. \qquad (21)$$

If the heat flux has a constant value at wall, the following equation can be obtained:

$$\frac{\partial T_{q\infty}}{\partial x} = \frac{\partial T_m}{\partial x} = -\frac{q}{a} \frac{1}{\rho c_p u_m}. \qquad (22)$$

($q > 0$ for heat release from the fluid).

Then, $T_{q\infty}$ yields

$$T_{q\infty} = -\frac{q}{a} \frac{x}{\rho c_p u_m} + G(y). \qquad (23)$$

By substituting this equation into equation (21), the following relation is obtainable:

$$-\frac{u}{u_m} \frac{q}{a \rho c_p} = k \frac{d^2G}{dy^2}.$$

Hence,

$$G(\eta) = -\frac{qa}{\lambda}\left[\frac{(\eta^2/2)\cosh K - (1/K^2)\cosh(K\eta)}{\cosh K - (1/K)\sinh K} + D_o\right]. \tag{24}$$

In order to determine the temperature in the thermal entrance region, the following temperature, T_q^+, is introduced:

$$T_q = T_q^+ + T_{q\infty}. \tag{25}$$

Substituting this into equation (7) and applying equation (23), the following relation is obtainable:

$$u\frac{\partial T_q^+}{\partial x} = k\frac{\partial^2 T_q^+}{\partial y^2}. \tag{26}$$

Boundary conditions:

$$x = 0:\ T_q^+ = 0,\ y = 0,\ y = a:\ \partial T_q^+/\partial y = 0.$$

Since this differential equation and boundary conditions are identically equal to equation (15), the above mentioned eigenvalues and eigenfunctions are also valid in this case. Thus,

$$T_q^+ = \frac{qa}{\lambda}\sum_{n=1}^{\infty} D_n Y_n \exp\left(-\frac{u_m}{u_{\max}}\beta_n\xi\right). \tag{27}$$

From the characteristics of Sturm–Liouville differential equation and the condition of $T_q^+ = 0$ at $x = 0$, the coefficients D_n can be calculated by the following relations,

$$\left.\begin{aligned} D_n &= -\frac{\int_0^1 u\left[\frac{G(\eta)}{qa/\lambda} - D_o\right] Y_n\,d\eta}{\int_0^1 u Y_n^2\,d\eta} \\ D_o &= -\frac{\int_0^1 u\left[\frac{G(\eta)}{qa/\lambda} - D_o\right] d\eta}{\int_0^1 u\,d\eta}. \end{aligned}\right\} \tag{28}$$

An example of the fluid temperature distribution is illustrated for $K = 4$ in Fig. 3.

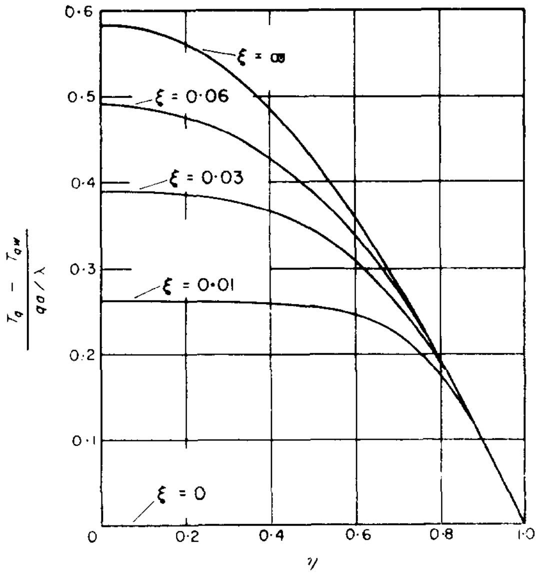

FIG. 3. Fluid temperature distribution with uniform heat transfer through the wall and no Joule heating ($K = 4$).

2.1.3. *Joule heating with wall heat transfer (uniform wall heat flux).* By using equation (8), the resultant temperature can be calculated as follows:

$$T = \left(\frac{1}{au_m c_p\rho}\int_0^a \frac{j_z^2}{\sigma}\,dy - \frac{q}{a}\frac{1}{u_m c_p\rho}\right)x + F(\eta) + G(\eta) + \frac{(Re\,Pr)^2\nu k}{a^2 c_p}\sum_{n=1}^{\infty} C_n Y_n \exp\left(-\frac{u_m}{u_{\max}}\beta_n\xi\right) + \frac{qa}{\lambda}\sum_{n=1}^{\infty} D_n Y_n \exp\left(-\frac{u_m}{u_{\max}}\beta_n\xi\right). \tag{29}$$

In Fig. 4, the temperature distributions for $K = 4$ and 8 at $\xi = 0$, 0·03 and infinity are illustrated. In this figure, M is a parameter which is given by the following formula:

$$M = \frac{(Re\,Pr)^2\nu k\lambda}{a^2 c_p qa}. \tag{30}$$

2.2. *Uniform wall temperature*

If the inlet temperature at $x = 0$ is denoted by T_o and the wall temperature is assumed to be equal to zero, the problem is divided into two simpler problems as follows:

(3) A case having Joule heating and zero inlet temperature.

(4) A case having no Joule heating but an arbitrary inlet temperature T_o.

In this section, the fluid temperature is

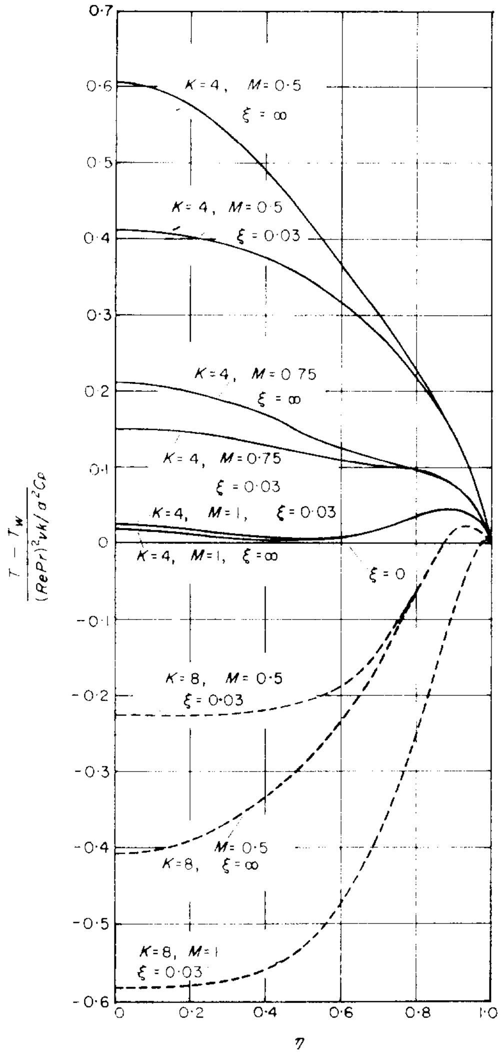

FIG. 4. Fluid temperature distribution in the thermal entrance region under uniform heat flux at wall.

defined as the excess temperature over the value at wall. If each solution can be written as T_H and T_h, respectively, the solution of general problem yields

$$T = T_H + T_h. \tag{31}$$

And the equations for problem (3) and (4) are as follows:

(3)
$$u\frac{\partial T_H}{\partial x} = k\frac{\partial^2 T_H}{\partial y^2} + \frac{j_z^2}{\sigma}\frac{1}{c_p\rho}. \tag{32}$$

Boundary conditions:

$$x = 0:\ T_H = 0,\ y = 0:\ \partial T_H/\partial y = 0,\ y = a:\ T_H = 0.$$

(4)
$$u\frac{\partial T_h}{\partial x} = k\frac{\partial^2 T_h}{\partial y^2}. \tag{33}$$

Boundary conditions:

$$x = 0:\ T_h = T_o,\ y = 0:\ \partial T_h/\partial y = 0,\ y = a:\ T_h = 0.$$

2.2.1. *Zero inlet temperature with Joule heating.* Considering the fully developed region, since the temperature profile is kept constant pattern, the fluid becomes isothermal in x-direction. Then, equation (32) can be transformed into

$$k\frac{\partial^2 T_{H\infty}}{\partial y^2} + \frac{j_z^2}{\sigma}\frac{1}{c_p\rho} = 0. \tag{34}$$

This equation can be integrated as follows:

$$T_{H\infty} = -\frac{(Re\,Pr)^2\nu k}{a^2 c_p}\left(\frac{\sinh K}{\cosh K - (1/K)\sinh K}\right)^2 \left\{\frac{\eta^2}{2} - \frac{2\cosh(K\eta)}{K\sinh K} + \frac{K^2}{2\sinh^2 K}\left[\frac{\eta^2}{2} + \frac{1}{4K^2}\cosh(2K\eta)\right] - \frac{1}{2} + \frac{2\cosh K}{K\sinh K} - \frac{K^2}{2\sinh^2 K}\left[\frac{1}{2} + \frac{1}{4K^2}\cosh(2K)\right]\right\}. \tag{35}$$

To determine the thermal entrance region, a temperature T_H^+ is introduced,

$$T = T_H^+ + T_{H\infty}. \tag{36}$$

Then, the fundamental equation (32) is transformed into

$$u\frac{\partial T_H^+}{\partial x} = k\frac{\partial^2 T_H^+}{\partial y^2}. \tag{37}$$

Boundary conditions:

$$y = 0:\ \partial T_H^+/\partial y = 0,\ y = a:\ T_H^+ = 0.$$

The solution of this equation can be expressed as follows:

$$T_H^+ = \frac{(Re\,Pr)^2\nu k}{a^2 c_p}\sum_{n=1}^{\infty} E_n\psi_n \exp\left(-\frac{u_m}{u_{\max}}\gamma_n\xi\right), \tag{38}$$

where γ_n and ψ_n are the eigenvalues and eigenfunctions, respectively, of the following Sturm–Liouville type differential equation,

$$\frac{d^2\psi_n}{d\eta^2} + \gamma_n \frac{u}{u_{max}} \psi_n = 0. \tag{39}$$

Boundary conditions:

$$\eta = 0:\ d\psi_n/d\eta = 0, \eta = 1:\ \psi_n = 0.$$

And the coefficients E_n are expressed as follows by the characteristics of Sturm–Liouville equation and the condition of $T_H^+ = 0$ at $x = 0$:

$$E_n = -\frac{\int_0^1 u \frac{T_{H\infty}}{[(Re\,Pr)^2 \nu k]/a^2 c_p} \psi_n\, d\eta}{\int_0^1 u\psi_n^2\, d\eta}. \tag{40}$$

These numerical calculations were done by the digital computer KDC-1. The eigenvalues for several kinds of K are tabulated in Table 2. In the case of $K = 0$ which means Newtonian flow, γ_n agrees well with the values by Sellars *et al.* [7]. Fig. 5 shows the temperature distribution at several kinds of ξ.

Table 2. Eigenvalues

	n	Authors' values	Sellars' values
K=0	1	2·8447	2·779
	2	32·3546	32·11
	3	94·0990	93·45
	4	188·1117	186·90
	5	314·3100	312·20
K=4	1	2·6390	
	2	27·1269	
	3	78·3049	
	4	156·2148	
	5	261·0073	
K=8	1	2·5201	
	2	23·9883	
	3	68·6067	
	4	136·5288	
	5	227·6907	

2.2.2. *Arbitrary inlet temperature without Joule heating.* Since the fluid flows with wall heat transfer under the condition of $T_w = 0$, the fluid temperature, $T_{h\infty}$, in the fully developed region becomes $T_{h\infty} = 0$. And introducing the temperature, T_h^+, in the thermal entrance region,

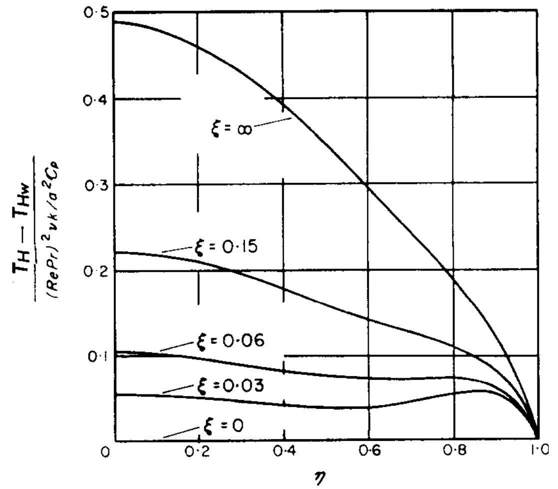

FIG. 5. Fluid temperature distribution with Joule heating and the same inlet temperature as the wall ($K = 4$).

$$T_h = T_h^+ + T_{h\infty}, \tag{41}$$

the fundamental equation (33) yields

$$u\frac{\partial T_h^+}{\partial x} = k\frac{\partial^2 T_h^+}{\partial y^2}. \tag{42}$$

Boundary conditions:

$$y = 0:\ \partial T_h^+/\partial y = 0, y = a:\ T_h^+ = 0.$$

Since this equation has the same formula and the same boundary conditions as equation (37), T_h^+ can be expressed as the infinite series by using the above mentioned eigenvalues γ_n and eigenfunctions ψ_n.

$$\frac{T_h^+}{\frac{(Re\,Pr)^2 \nu k}{a^2 c_p}} = \frac{T_o}{\frac{(Re\,Pr)^2 \nu k}{a^2 c_p}} \sum_{n=1}^{\infty} F_n \psi_n \exp\left(-\frac{u_m}{u_{max}}\gamma_n \xi\right). \tag{43}$$

And the coefficients F_n are given by considering the boundary condition, $T_h^+ = T_o$ at $x = 0$,

$$F_n = \frac{\int_0^1 u\psi_n\, d\eta}{\int_0^1 u\psi_n^2\, d\eta}. \tag{44}$$

An example of the temperature distribution is shown in Fig. 6.

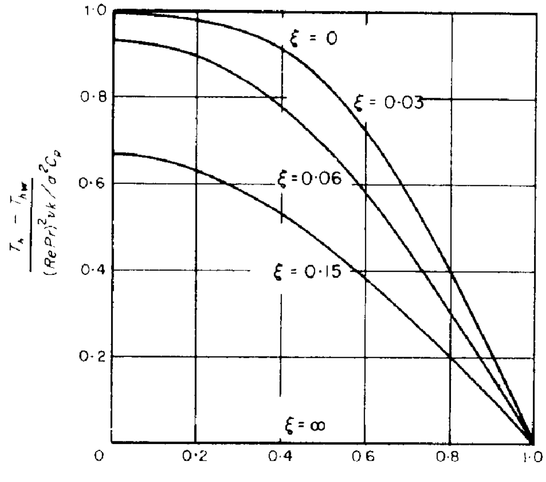

FIG. 6. Fluid temperature distribution with no Joule heating under $(T_o - T_w)/(Re\,Pr)^2 \nu k/a^2 cp = 1.$ $(K = 4)$.

2.2.3. *Joule heating with wall heat transfer (uniform wall temperature).* From equation (31), the resultant temperature can be written as follows:

$$T = T_{H\infty} + \frac{(Re\,Pr)^2 \nu k}{a^2 c_p} \sum_{n=1}^{\infty} E_n \psi_n \exp\left(-\frac{u_m}{u_{\max}} \gamma_n \xi\right) + T_o \sum_{n=1}^{\infty} F_n \psi_n \exp\left(-\frac{u_m}{u_{\max}} \gamma_n \xi\right). \tag{45}$$

The fluid temperature distribution calculated is shown in Fig. 7.

3. NUSSELT NUMBER

Now consider the following heat-transfer coefficient h,

$$h = \frac{q}{T_m - T_w}. \tag{46}$$

And local Nusselt number Nu is defined as follows:

$$Nu = \frac{h(2a)}{\lambda} = \frac{2a}{\lambda} \frac{q}{T_m - T_w} = -2a \frac{(\partial T/\partial y)_{y=a}}{T_m - T_w}. \tag{47}$$

In the case of uniform wall heat flux, the mixed-mean temperature T_m is

$$T_m = \left(\frac{1}{a u_m c_p \rho} \int_0^a \frac{j_z^2}{\sigma} \mathrm{d}y - \frac{q}{a} \frac{1}{u_m c_p \rho}\right) x. \tag{48}$$

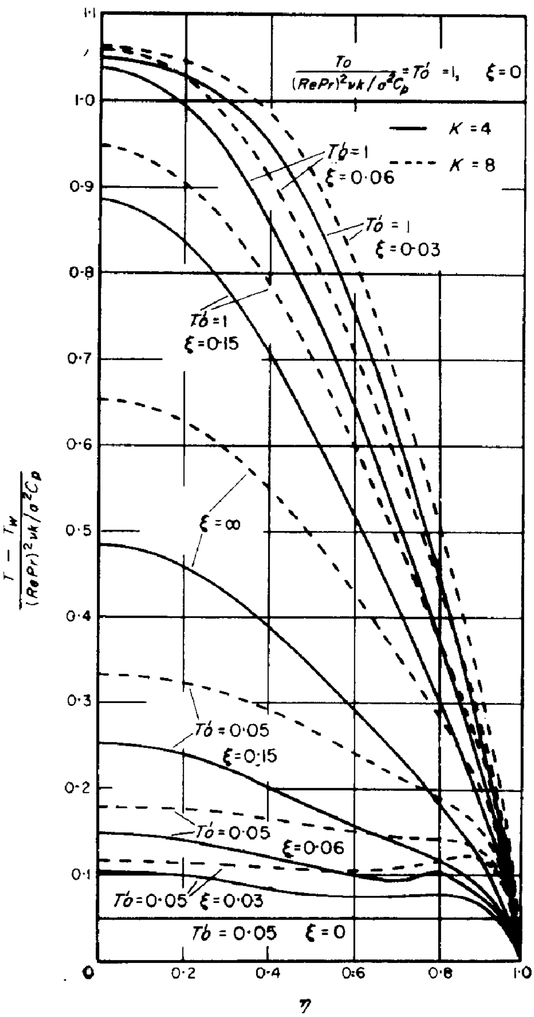

FIG. 7. Fluid temperature distribution in thermal entrance region under uniform wall temperature.

And $T_m - T_w$ is given by the following equation,

$$T_m - T_w = -F(1) - G(1) - \frac{(Re\,Pr)^2 \nu k}{a^2 c_p} \sum_{n=1}^{\infty} C_n Y_n(1) \exp\left(-\frac{u_m}{u_{\max}} \beta_n \xi\right) - \frac{qa}{\lambda} \sum_{n=1}^{\infty} D_n Y_n(1) \exp\left(-\frac{u_m}{u_{\max}} \beta_n \xi\right). \tag{49}$$

Consequently, Nusselt number Nu_1 becomes

$$Nu_1 = - \frac{2qa/\lambda}{F(1) + G(1) + \frac{(Re\,Pr)^2 \nu k}{a^2 c_p} \sum_{n=1}^{\infty} C_n Y_n(1) \exp\left(-\frac{u_m}{u_{\max}} \beta_n \xi\right) + \frac{qa}{\lambda} \sum_{n=1}^{\infty} D_n Y_n(1) \exp\left(-\frac{u_m}{u_{\max}} \beta_n \xi\right)}. \tag{50}$$

If the Joule heating is neglected, the Nusselt number Nu_1 is

$$Nu_1' = - \frac{2qa/\lambda}{G(1) + \frac{qa}{\lambda} \sum_{n=1}^{\infty} D_n Y_n(1) \exp\left(-\frac{u_m}{u_{\max}} \beta_n \xi\right)}. \tag{51}$$

In the case of uniform wall temperature, $T_m - T_w$ is

$$T_m - T_w = \frac{u_{\max}}{u_m} \left\{ \frac{(RePr)^2 \nu k}{a^2 c_p} \sum_{n=1}^{\infty} E_n \left[1 - \exp\left(-\frac{u_m}{u_{\max}} \gamma_n \xi\right)\right] \frac{\psi_n'(1)}{\gamma_n} + T_o \sum_{n=1}^{\infty} F_n \exp\left(-\frac{u_m}{u_{\max}} \gamma_n \xi\right) \frac{\psi_n(1)}{\gamma_n} \right\} \tag{52}$$

and

$$q = -\lambda(\partial T/\partial y)_{y=a}. \tag{53}$$

Hence, Nusselt number Nu_2 is

$$Nu_2 = 2\frac{u_m}{u_{\max}} \left\{ \sum_{n=1}^{\infty} E_n \psi_n'(1) \left[1 - \exp\left(-\frac{u_m}{u_{\max}} \gamma_n \xi\right)\right] - \frac{T_o}{(Re\,Pr)^2 \nu k / a^2 c_p} \sum_{n=1}^{\infty} F_n \psi_n'(1) \exp\left(-\frac{u_m}{u_{\max}} \gamma_n \xi\right) \right\} \Big/ \left\{ \sum_{n=1}^{\infty} E_n \frac{\psi_n'(1)}{\gamma_n} \left[1 - \exp\left(-\frac{u_m}{u_{\max}} \gamma_n \xi\right)\right] - \frac{T_o}{(Re\,Pr)^2 \nu k / a^2 c_p} \sum_{n=1}^{\infty} F_n \frac{\psi_n'(1)}{\gamma_n} \exp\left(-\frac{u_m}{u_{\max}} \gamma_n \xi\right) \right\}. \tag{54}$$

If the Joule heating is neglected, Nusselt number Nu_2' is

$$Nu_2' = 2\frac{u_m}{u_{\max}} \frac{\sum_{n=1}^{\infty} F_n \psi_n'(1) \exp\left(-\frac{u_m}{u_{\max}} \gamma_n \xi\right)}{\sum_{n=1}^{\infty} F_n \frac{\psi_n'(1)}{\gamma_n} \exp\left(-\frac{u_m}{u_{\max}} \gamma_n \xi\right)}. \tag{55}$$

Fig. 8 shows the relation between local Nusselt number and ξ, in the case of no Joule heating. In this figure, the full lines show the uniform heat flux and the dotted lines the uniform wall temperature. Moreover, the case of $K = 0$ is equivalent to non-conducting Newtonian fluid flow and the calculated values coincide well with Sellars *et al.* [7]. Fig. 9 represents the relation between local Nusselt number and ξ under uniform heat flux. Fig. 10 shows the case of uniform wall temperature. As compared with these figures, it should be noted that the Joule

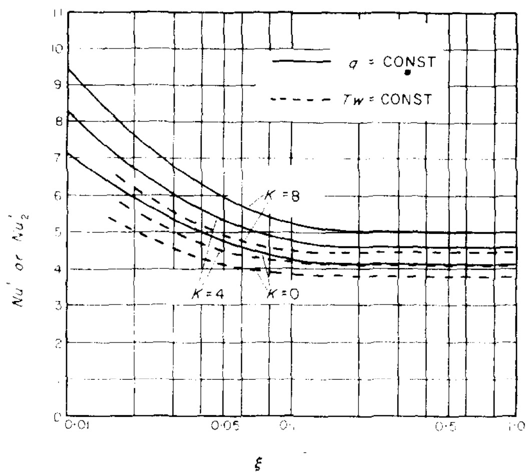

FIG. 8. Local Nusselt number in the case of no Joule heating.

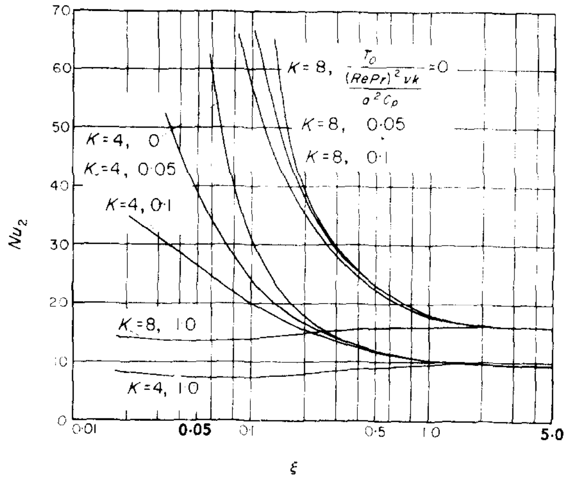

FIG. 10. Local Nusselt number under uniform wall temperature.

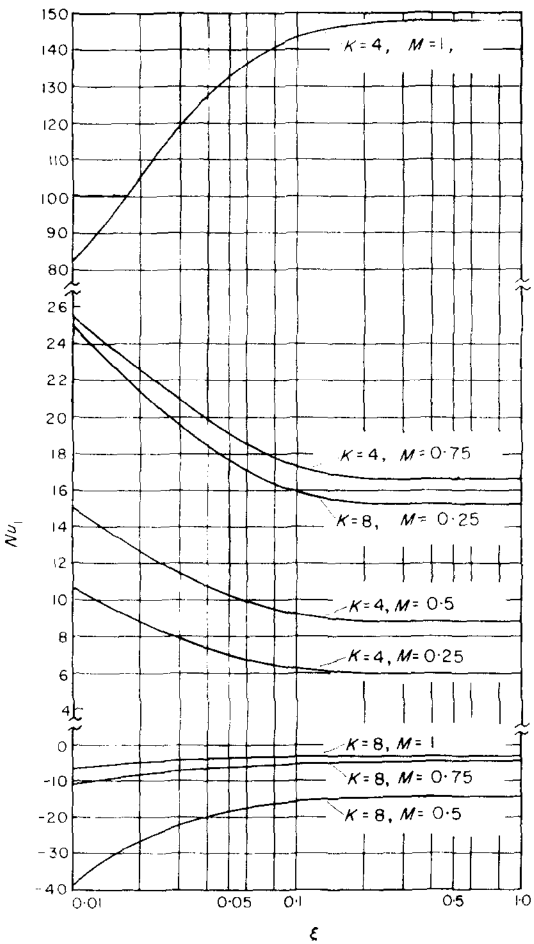

FIG. 9. Local Nusselt number under uniform heat flux.

heating affects so severely the heat transfer that it must not be ignored. Of course, the effects of K, M and $T_0/[(Re\ Pr)^2 \nu k/a^2 c_p]$ on the heat transfer must be taken into consideration.

4. CONCLUSION

An analysis was made to determine the heat transfer by Hartmann's flow under the conditions of uniform wall heat flux and uniform wall temperature. Though it is necessary to calculate numerous eigenvalues and eigenfunctions in order to analyse more precisely the heat transfer near the inlet, the results obtained in this paper will be undoubtedly applicable to the convective heat transfer in the thermal entrance region.

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APPENDIX

Equation (19) is rewritten here,

$$\frac{d^2 Y_n}{d\eta^2} + \beta_n \frac{u}{u_{\max}} Y_n = 0. \tag{19}$$

Instead of the above equation, Siegel *et al.* used the following equation:

$$\frac{d^2 Y_n}{d\eta^2} + \frac{2}{3}\beta_{n\ \text{Siegel}} \frac{u}{u_m} Y_n = 0. \tag{56}$$

Thus,

$$(\beta_{n\ \text{Siegel}}) \left(\frac{2}{3}\frac{u_{\max}}{u_m}\right) = \beta_n'. \tag{57}$$

the values of β_n' are tabulated in Table 1.

Résumé—Cet article concerne une étude théorique de la détermination des caractéristiques des échanges thermiques dans la zone d'entrée (mise en équilibre thermique) dans le cas d'un écoulement laminaire de fluide, électriquement conducteur, entre deux plaques parallèles, avec échange de chaleur à la paroi, et soumis à un champ magnétique transversal; cet écoulement est appelé écoulement de Hartmann.

Les conditions d'échanges thermiques à la paroi considérés ici sont à la fois celles correspondant à un flux de chaleur pariétal uniforme et celles d'une température de paroi uniforme. On tient compte dans cette étude de la production d'énergie interne par effet Joule, mais on néglige la dissipation visqueuse.

La méthode analytique utilisée est analogue à celle choisie par les autres auteurs.

Les caractéristique thermiques de la région de mise en équilibre se rapprochent de celles étudiées par Siegel pour le régime établi. Les échanges thermiques dans le cas de l'écoulement d'un fluide non conducteur, étudié en tant que cas particulier, coïncident bien avec les résultats de Sellar et des autres.

Ces calculs numériques ont été faits sur la calculatrice KDC-1 de l'Université de Kyoto.

Zusammenfassung—Die Arbeit umfasst eine theoretische Analyse zur Bestimmung der Wärmeübergangscharakteristik im thermischen Anlaufbereich einer elektrisch leitenden Flüssigkeit bei sogenannter Hartmannströmung. Das Medium strömt dabei, laminar ausgebildet, zwischen zwei parallelen Platten, an welchen, unter dem Einfluss eines einheitlichen magnetischen Querfeldes, ein Wärmeübergang erfolgt. Der Wärmeübergang ist sowohl für den Fall gleichbleibender Wärmestromdichte von der Wand, als auch konstanter Wandtemperatur untersucht. Die innere Joule'sche Wärmeerzeugung ist in der Analyse berücksichtigt, nicht jedoch die viskose Dissipation. Die analytische Methode ist analog der vom Autor in anderen Arbeiten benutzten.

Die Charakteristika für den Wärmeübergang im Einlaufgebiet nähern sich jenen der voll ausgebildeten Strömung, die Siegel analysierte. Der Wärmeübergang bei laminarer Strömung eines nichtleitende Mediums, der als Spezialfall analysiert wurde, stimmt gut mit Ergebnissen von Sellars und anderen überein.

Die numerischen Rechnungen wurden mit Hilfe des KDC-1 (Kyoto University Digital Computer-1) ausgeführt.

Аннотация—В статье дается теоретический анализ определения характеристик переноса на входном участке нагрева при установившемся ламинарном течении электропроводной жидкости между двумя параллельными пластинами, когда перенос тепла на стенке происходит под влиянием постоянного поперечного поля (так называемое течение Хартманна). В статье рассматриваются следующие условия переноса тепла на стенке: при заданном постоянном тепловом потоке на стенке и при постоянной температуре стенки. Анализ ведется с учётом джоулева тепла. Вязкой диссипацией пренебрегается. Аналитический метод аналогичен методу, используемому автором в другой его работе.

Характеристики переноса тепла на входном участке близки к характеристикам полностью развитого течения-случай, рассмотренный Сиджелем. Перенос тепла при ламинарном течении неэлектропроводной жидкости, рассмотренный как частный случай, совпадает со случаями, проанализированными Селларсом и другими.

Эти численные расчёты производились с помощью КДС-1 (цифровой вычислительной машины Университета Киото модели 1).